Terr. Atmos. Ocean. Sci., Vol. 23, No. 4, 451-458, August 2012 doi: 10.3319/TAO.2012.02.14.01(Hy)

Retention Characteristics for Multiple-PhaseFluid Systems

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Received 11 October 2011, accepted 14 February 2012

ABSTRACT

The key objective of this paper is to point out that a widely used constitutive relationship between the degrees of saturation and capillary pressures for three-phasefluids over the past twenty five years has resulted in an undue constraint. To our knowledge, this constraint is neither physically justifiable nor is it theoretically supported. The discovery of the undue constraint leads this investigation to develop a viable parametric model to describe the constitutive relationship for a system of multiple fluids. Based on the physical consideration wherein fluid wettability follows a sequence, the proposed parametric model can be easily applied to any system of L arbitrary fluids. Three aspects are presented to confirm the plausibility and completeness of the model. First, the proposed closed-form expression of the saturation-capillary pressure head relationship is identical to van Genuchten's renowned model for two-phasefluid systems. Second, a constraint appearing in the widely used model is alleviated in the proposed model. Third, seven hypothetical examples are used to demonstrate that there is no need to impose the constraint on the sequence of capillary pressures. The results show the plausibility and completeness of the model for systems of multiple fluids. The proposed parametric model is a feasible analytic model which provides a generalized saturation-capillary pressure head relationship for any system with regard to multiple fluids. Hopefully, this study will pave a way for others to conduct experiments to validate the model further or develop other better non-constrained models based on experimental evidence.

Key words: Multiple-phasefluids, Constitutive relation, van Genuchten's model, Saturation-capillary pressure head relationship

Citation: Tsai, C. H. and G. T. Yeh, 2012: Retention characteristics for multiple-phasefluid systems. Terr. Atmos. Ocean. Sci., 23, 451-458, doi: 10.3319/TAO.2012.02.14.01(Hy)

1. INTRODUCTION

In general, when the number of fluids for multiple-phasefluid systems exceeds two, it becomes difficult to simultaneously measure the degrees of saturation and capillary pressure heads between two contacting fluids. Therefore, developing a plausible analytic model to describe the saturation-capillary pressure head relationship is necessary for systems of multiple-phasefluids. Numerous parametric models have been presented to describe the constitutive relation of saturation and capillary pressure head in vadose zones (e.g., Brooks and Corey 1964; Campbell 1974; van Genuchten 1980). Among them, the renowned model by van Genuchten (1980) is one of the most popular models to describe the saturation-capillary pressure head relationship in unsaturated soil systems. Since the model was originally proposed for vadose zones (water-air systems), it is limited to the description of the relationship of two-phasefluid systems, such as water-air, water-non aqueous phase liquid (NAPL) and NAPL-air systems. Fundamental to all of these models is the hypothesis that the capillary pressure is a function of the degree of saturation of the wetting phase.

However, extending the above hypothesis to multiple-phasefluid systems is not straightforward. First, when there are more than two-phasefluids in a system, there are more than one independent degrees of saturation. For example, in a multiple-phasefluid system with L phases, there are L degrees of saturation, but only (L-1) of these degrees of saturation are independent because the total degree of saturation is equal to one. Second, there are (L-1) interfaces, each between two contacting fluids; hence, there are (L-1) capillary pressures and each represents the pressure difference between two consecutive contacting fluids. In the most general case, each capillary pressure is a function of (L-1)

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degrees of saturation. In this paper, a homogeneous formulation of capillary pressure is hypothesized based upon a careful examination of Parker et al.'s model (1987) which is an extension of van Genuchten's two-phasefluid model (1980) to the three-phasefluid system.

The extension of van Genuchten's equation to three-phasefluids by Parker et al. (1987) was developed based on the assumption that the fluid wettability follows the sequence: water → NAPL → air. Since the closed-form expression of the saturation-capillary pressure head relationship is quite simple, the model has been widely used to describe the relationships between the degrees of saturation and capillary pressure heads in two- and three-phasefluid systems (e.g., Parker and Lenhard 1987; Binning and Celia 1999; Suk and Yeh 2008; Khoei and Mohammadnejad 2011). Based on a careful examination of this extension, it is found that an undue constraint exists in the extension model (Parker and Lenhard 1987) - that the product of scaling factor and capillary pressure head between water and NAPL must be greater than that between NAPL and air. From the viewpoint of physics, this constraint has not been supported experimentally or theoretically. To our knowledge, this unproven constraint has not been discussed in literature.

In this investigation, a parametric model is proposed to provide a plausible constitutive relation of saturation and capillary pressure head for a system of more than two-phasefluids. The concept is based mainly on the physical consideration that (1) L-phasefluids follow the sequence of wettability, and (2) at the interface face of two fluids, all fluids on one side of the sequence are considered to be a relatively wetting phase and all fluids on the other side are considered to be a relatively non-wetting phase. With this concept, the saturation-capillary pressure relationships between two contacting phasefluids are postulated to be a unique function of a single variable of grouped degrees of saturation. An explicit function is proposed that is analogous to van Genuchten's model. Through mathematical and physical examinations, the undue constraint resulting from the model proposed by Parker et al. (1987) is alleviated. In addition, note that the proposed closed-form expression of the saturation-capillary pressure relationship is reduced to that of the model by van Genuchten (1980) for two-phase-fluid systems in formality.

The remainder of this paper is organized as follows. In section 2, a fundamental postulation is posed for describing the capillary pressures as functions of degrees of saturation in systems of multiple-phasefluids. In section 3, two widely used parametric models (van Genuchten 1980; Parker et al. 1987) are discussed, and potential problems are acknowledged. In section 4, a generalized closed-form expression of the saturation-capillary pressure relationship for a system of L-phasefluids is proposed like that of van Genuchten's model. In section 5, seven hypothetical examples are employed to demonstrate the plausibility and completeness of the proposed model. Conclusions are offered in section 6.

2. SATURATION-CAPILLARY PRESSURE FUNCTIONS

The Young-Laplace equation (Laplace 1806) is used to describe the capillary pressure sustained across the interface of two immiscible fluids, such as water and air, due to the phenomena of surface tension (Fig. 1). The equation can be derived based on the force balance on a surface element or on the concept of virtual work to extend the surface (Skjæveland 2006). The equation (Laplace 1806) is given as

$$p_c = \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad \text{or} \quad p_c = \frac{2\sigma}{r_c} \quad \text{when } r_1 = r_2 = r_c \tag{1}$$

where $p_c \equiv p_2 - p_1$ is the capillary pressure, with p_1 being the pressure of wetting phase-fluid 1 and p_2 being the pressure of non-wetting phase-fluid 2, $[\mathrm{M}/(\mathrm{T}^2\mathrm{L})]$; σ is the surface tension, $[\mathrm{M}/(\mathrm{T}^2)]$; r_1 and r_2 are the principal radii of curvature, [L]; and r_c is the radius of curvature, [L].

In hydrology communities dealing with the two-phase-fluid system in subsurface porous media, the radius of curvature is indexed by the pore size occupied by the wetting phasefluid, i.e., the radius of curvature is represented by the degree of saturation of the wetting phase. The degree

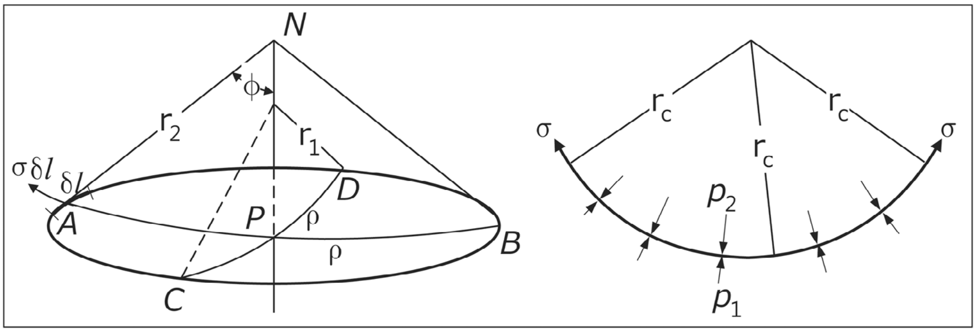

Fig. 1. Capillary pressure, surface tension, and radius of curvature.

of saturation of non-wetting phasefluid does not play a role because it is not an independent variable (the degree of saturation of wetting phasefluid plus that of non-wetting phasefluid is equal to one). Thus, it has been postulated that the capillary pressure for two-phasefluids is a function of the degree of saturation:

$$p_2 - p_1 \equiv p_c = f(S) \tag{2}$$

where S is the degree of saturation, and the capillary pressure function f has the following properties: (1) domain of the function, S, varies from zero to one; (2) the range of the function, p_c, varies from zero to infinity; and (3) $p_c = \infty$ when $S = 0$ and $p_c = p_b$ when $S = 1$. The saturation function is obtained by inverting Eq. (2) as

$$S = f^{-1}(p_c) = F(p_c) \tag{3}$$

When there are more than two-phasefluids in a system, says L, there are L degrees of saturation and (L-1) capillary pressures, each being the pressure difference between the pressures of a pair of contacting fluids (Fig. 2). Among the L degrees of saturation, only (L-1) of these, says S_1, S_2,..., and S_{L-1}, are independent because the summation of all L degrees of saturation is equal to one. In the most general cases, the capillary pressures between any two consecutive phasefluids would be a function of all (L-1) independent degrees of saturation. Before proceeding further, we define the accumulated degree of saturation as

$$S_{t\alpha} = \sum_{\beta=1}^{\alpha} S_\beta \quad \text{for} \quad \alpha = 1, 2, ..., L; \quad \text{Note: } S_{tL} = 1 \text{ and } S_{t0} = 0 \tag{4}$$

where $S_{t\alpha}$ is the total degree of saturation accumulated up to the α-th phase (Fig. 2). Note that the fluid phases are labeled consecutively starting from the most wetting phase to the most non-wetting phase.

In terms of the total accumulated degrees of saturation, the most general capillary pressure function between any two consecutive phases can be posited as

$$\begin{aligned} p_{\alpha+1} - p_\alpha &\equiv p_{c\alpha+1,\alpha} = \\ & f_\alpha(S_{t\alpha+1}, S_{t\alpha}, S_{t\alpha-1}, ..., S_{t\alpha+1-k}) \quad \text{for} \quad \begin{cases} \alpha = 1, 2, ..., L-1 \\ k = 1, 2, ..., \alpha \end{cases} \end{aligned} \tag{5}$$

where $p_{\alpha+1}$ is the pressure of the $(\alpha+1)$-th phase, $[M/(T^2L)]$; p_α is the pressure of the α-th phase, $[M/(T^2L)]$; $p_{c\alpha+1,\alpha}$ is the capillary pressure between the $(\alpha+1)$-th and α-th phases, $[M/(T^2L)]$; f_α is the capillary pressure function between the relative wetting phase α and the relative non-wetting phase $\alpha+1$. We first hypothesize that (1) the capillary pressure function is homogeneous, i.e., f_α is independent of α; and (2) the capillary pressure function f_α is a function of $S_{t\alpha}$ and $S_{t\alpha+1}$ only, i.e., the capillary pressure function f_α is of degree 1, aka (as known as) $k = 1$. Then we postulate that the capillary pressure $p_{c\alpha+1,\alpha}$ is a unique function of a single variable defined as the ratio of the two total accumulated degrees of saturation as

$$\begin{aligned} & p_{\alpha+1} - p_\alpha \equiv p_{c\alpha+1,\alpha} = f(\Theta_\alpha) \quad \text{for} \quad \alpha = 1, 2, ..., L-1 \\ & \qquad \text{in which} \quad \Theta_\alpha \equiv \frac{S_{t\alpha}}{S_{t\alpha+1}}; \ \Theta_L = 1; \text{ and } \Theta_0 = 0 \end{aligned} \tag{6}$$

where Θ_α is the ratio of the total accumulated degree of saturation of the relatively wetting phase to that of the relatively non-wetting phase. Inverting Eq. (6), we obtain

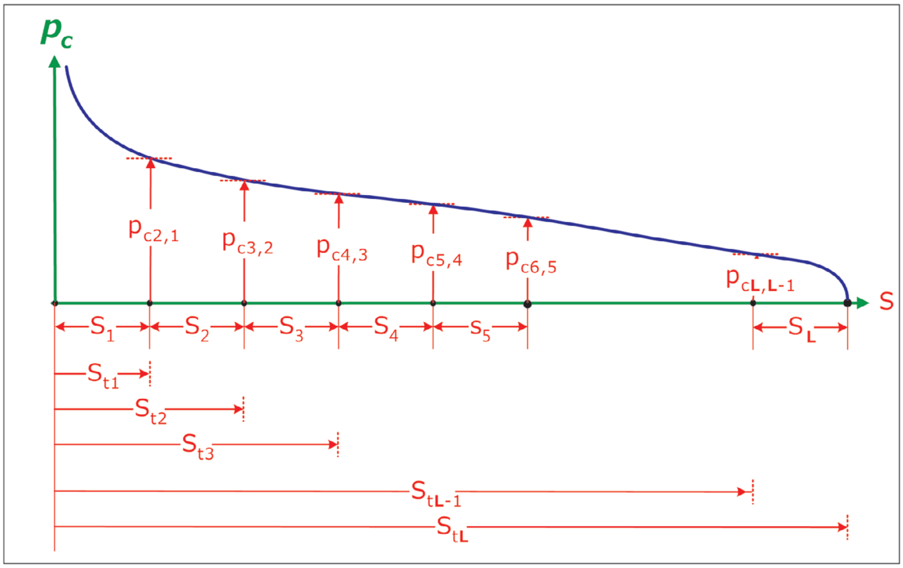

Fig. 2. Sketch of capillary pressures versus degrees of saturation for an L-phasefluids system.

$$\Theta_\alpha = f^{-1}(p_{c\alpha+1,\alpha}) = F(p_{c\alpha+1,\alpha}),\ \alpha = 1, 2, ..., L-1;\ \Theta_L = 1;\ \Theta_0 = 0 \tag{7}$$

It is easily seen that Eqs. (6) and (7) are the generalizations of Eqs. (2) and (3), respectively. Clearly, for two-phase-fluids, $S_{t1} = S$ and $S_{t2} = 1$. Thus, under this case, Eqs. (6) and (7) are reduced to Eqs. (2) and (3), respectively.

3. TWO WIDELY USED PARAMETRIC MODELS

3.1 van Genuchten's Model for Two-Phasefluids

Many explicit functions based on the postulation of Eq. (2) have been proposed to relate the degree of saturation to the capillary pressure (Charbeneau 2000) for two- phasefluids. For example, the renowned model by van Genuchten (1980) is given as

$$S = 1 \text{ for } p_c \le 0 \quad \text{and} \quad S = [1 + (\hat{\alpha} p_c)^N]^{-M} \text{ for } p_c > 0 \tag{8}$$

where $\hat{\alpha}$ is the scaling factor, N is the curve shape parameter, and $M = 1 - 1/N$. Written in terms of a capillary pressure head, van Genuchten's model (van Genuchten 1980) in Eq. (8) is expressed analytically as

$$S = 1 \text{ for } h_c \le 0 \quad \text{and} \quad S = [1 + (\overline{\alpha} h_c)^N]^{-M} \text{ for } h_c > 0 \tag{9}$$

$\overline{\alpha}$ is the scaling factor of the capillary pressure head (Charbenuar 2000). Note that van Genuchten's model was originally proposed to describe the saturation-capillary pressure head relationship for two phase-fluids. For example, in a water-air system, both air and water are present in the system when $0 < S < 1$. Only water is present in the system when $S = 1.0$; and, only air is present in the system when $S = 0.0$.

An amplification is necessary before we describe a model for three-phasefluids in the following subsection. Mathematically, when the capillary pressure head $h_c = h_a - h_w$ (where h_a is the pressure head of non-wetting phase, air; and h_w is the pressure head of the wetting phase, water) is equal to zero in van Genuchten's model, the degree of saturation for the wetting phase is equal to one and that for the non-wetting phase is equal to zero. Physically, this means the non-wetting phasefluid, air, dissolves into the wetting phasefluid, water. In other words, the entrance capillary pressure in the water-air system is implicitly assumed to be zero in van Genechten's model. If the entrance capillary pressure is not approximately zero, other models may be used. For example, Brooks and Corey's model provides such an alternative (Brooks and Corey 1964).

$$S = 1 \text{ for } h_c \le h_b,\ S = \left(\frac{h_b}{h_c}\right)^{\lambda} \text{ for } h_c > h_b \tag{10}$$

where h_b is the bubble pressure or air entry pressure of the non-wetting phasefluid into wetting phasefluid, [L]; and λ is the pore size distribution index.

3.2 Parker et al.'s Model for Three-Phasefluids

Based on the assumption that fluid wettability follows the sequence water → NAPL → air, Parker et al. (1987) extended the saturation-capillary pressure head relationship in the model by van Genuchten (1980) from two-phasefluids to three-phasefluids. In their work (Parker et al. 1987), the three-phasefluids are water, NAPL and air, respectively. For consistence, the subscript 1 denotes water phasefluid, subscript 2 denotes NAPL phasefluid, and subscript 3 denotes air phasefluid following the wetting sequence. Thus, the degrees of saturation of water, NAPL and air are denoted as S_1, S_2, and S_3, respectively. In addition, two accumulated total saturations denoted by S_{t1} and S_{t2}, relating to S_1 and S_2, are given as follows

$$S_{t1} = S_1 \quad \text{and} \quad S_{t2} = S_1 + S_2 \tag{11}$$

With the definition of the accumulated liquid saturation, a naive extension of van Genuchten's model of two-phasefluids to three-phasefluids results in the following saturation-capillary pressure head relationship (Parker et al. 1987).

$$\begin{aligned} &S_1 \equiv S_{t1} = 1 \text{ for } h_{c21} \le 0 \quad \text{and} \quad S_1 \equiv S_{t1} = [1 + (\overline{\alpha}_{21} h_{c21})^N]^{-M} \\ &\hspace{20em} \text{for } h_{c21} > 0; \\ &S_t \equiv S_{t2} = 1 \text{ for } h_{c32} \le 0 \quad \text{and} \quad S_t \equiv S_{t2} = [1 + (\overline{\alpha}_{32} h_{c32})^N]^{-M} \\ &\hspace{20em} \text{for } h_{c32} > 0; \\ &S_2 = S_{t2} - S_{t1}; \quad \text{and} \quad S_3 = 1 - S_{t2} \end{aligned} \tag{12}$$

It is noted that $\overline{\alpha}_{\alpha+1,\alpha}$ is the scaling factor of capillary pressure head between the $(\alpha+1)$-th and α-th phases, $[1/\text{L}]$; and $h_{c\alpha+1,\alpha} \equiv h_{\alpha+1} - h_\alpha$ is the capillary pressure head between the $(\alpha+1)$-th and α-th phases, [L]; in which $h_{\alpha+1}$ is the pressure head of the $(\alpha+1)$-th phase, [L]; and h_α is the pressure head of the α-th phase, [L]. For example, h_{c32} and $\overline{\alpha}_{32}$ are the capillary pressure head and the scaling factor between the 3rd and 2nd phases, respectively. Similarly, h_{c21} and $\overline{\alpha}_{21}$ are, respectively, the capillary pressure head and the scaling factor between the 2nd and 1st phases. According to the definition of the total liquid saturation, S_{t2} is greater than or equal to S_{t1} as follows

$$S_{t2} \ge S_{t1} \tag{13}$$

Substituting Eq. (12) to Eq. (13), we have

$$[1 + (\overline{\alpha}_{32} h_{c32})^N]^{-M} \ge [1 + (\overline{\alpha}_{21} h_{c21})^N]^{-M} \tag{14}$$

which implies

$$(\overline{\alpha}_{32} h_{c32}) \leq (\overline{\alpha}_{21} h_{c21}) \qquad (15)$$

Obviously, Parker et al.'s model implies that the closed-form expression in Eq. (12) is workable only if the products of the scaling factor and capillary pressure head, $(\overline{\alpha}_{32} h_{c32})$ and $(\overline{\alpha}_{21} h_{c21})$, satisfy the constraint in Eq. (15). This constraint seems to have little physical relevance. In other words, some feasible distributions of a pressure head among phasefluids (that do not satisfy the constraint) are excluded. To our knowledge, there has been no literature to point out this undue phenomenon in three-phasefluids.

4. PROPOSED PARAMETRIC MODELS FOR MULTIPLE- PHASEFLUIDS

For a system of multiple-phasefluids, the degree of saturation function F in Eq. (7) is explicitly proposed in analogy to van Genuchten's model (van Genuchten 1980) as

$$\frac{S_{t\alpha}}{S_{t\alpha+1}} \equiv \Theta_{\alpha} = \begin{cases} 1 & \text{for } h_{c\alpha+1,\alpha} \leq 0 \\ \left[1 + (\overline{\alpha}_{c\alpha+1,\alpha}\, h_{c\alpha+1,\alpha})^{N}\right]^{-M} & \text{for } h_{c\alpha+1,\alpha} > 0 \end{cases}, \quad \alpha = 1, 2, ..., L-1;$$

$$\Theta_L = 1;$$

$$\Theta_0 = 0 \qquad (16)$$

Equation (7) could serve as a foundation to generalize other explicit functions for the degree of saturation in two-phasefluids to those in multiple-phasefluids. For example, Brooks and Corey's model can be extended to multiple-phasefluids as

$$\Theta_{\alpha} = \begin{cases} 1 & \text{for } h_{c\alpha+1,\alpha} \leq h_{b\alpha+1,\alpha} \\ \left(\dfrac{h_{b\alpha+1,\alpha}}{h_{c\alpha+1,\alpha}}\right)^{\lambda} & \text{for } h_{c\alpha+1,\alpha} > h_{b\alpha+1,\alpha} \end{cases}, \quad \alpha = 1, 2, ..., L-1;$$

$$\Theta_L = 1;$$

$$\Theta_0 = 0 \qquad (17)$$

where $h_{b\alpha+1,\,\alpha}$ is the bubble pressure head of the (α+1)-th non-wetting phasefluid into the α-th wetting phasefluid.

For a system of two-phasefluids, Eq. (16) is simplified to

$$\Theta = 1 \text{ for } h_c \leq 0 \quad \text{and} \quad \Theta = \left[1 + (\overline{\alpha} h_c)^{N}\right]^{-M} \text{ for } h_c > 0 \qquad (18)$$

Although Eq. (18) is identical to Eq. (9) as written, it has a very different physical interpretation. While in Eq. (9), S represents the degree of saturation of the wetting phase, Θ in Eq. (18) represents the grouped degree of saturation. The denominator for this grouped degree of saturation is equal to one, which renders the physical interpretation superficially identical. For any system with more than two-phasefluids, the denominator of the grouped degree of saturation is not necessarily equal to one, which makes the physical interpretation obviously different for a system of three-phasefluids.

For a system with three-phasefluids, the explicit function for the degree of saturation is straightforwardly obtained from Eq. (17) as

$$\Theta_1 \equiv \frac{S_{t1}}{S_{t2}} = 1 \text{ for } h_{c21} \leq 0 \quad \text{and} \quad \Theta_1 \equiv \frac{S_{t1}}{S_{t2}} = \left[1 + (\overline{\alpha}_{21} h_{c21})^{N}\right]^{-M} \text{ for } h_{c21} > 0;$$

$$\Theta_2 \equiv \frac{S_{t2}}{S_{t3}} = 1 \text{ for } h_{c32} \leq 0 \quad \text{and} \quad \Theta_2 \equiv \frac{S_{t2}}{S_{t3}} = \left[1 + (\overline{\alpha}_{32} h_{c32})^{N}\right]^{-M} \text{ for } h_{c32} > 0;$$

$$S_2 = S_{t2} - S_1; \quad \text{and} \quad S_3 = 1 - S_{t2} \qquad (19)$$

Examining Eq. (19), we see that both the numerator and denominator in the second line of Eq. (19) are greater than or equal to those in the first line of Eq. (19), respectively. Thus, it is not necessary that Θ_2 is greater than or equal to Θ_1. Hence, it is not necessary that $(\overline{\alpha}_{32} h_{c32})$ must be less than or equal to $(\overline{\alpha}_{21} h_{c21})$, which is in contrast to the consequence of that in Parker et al.'s model. The undue constraint on capillary pressures is therefore alleviated. Note that, based on Eq. (19), we obtain the following constraints

$$S_{t1} \leq S_{t2} \text{ and } S_{t2} \leq S_{t3} \rightarrow (\overline{\alpha}_{21} h_{c21}) \geq 0 \text{ and } (\overline{\alpha}_{32} h_{c32}) \geq 0, \text{ respectively} \qquad (20)$$

Unlike the constraint in Eq. (15) that does not have physical relevance, both constraints $(\overline{\alpha}_{21} h_{c21}) \geq 0$ and $(\overline{\alpha}_{32} h_{c32}) \geq 0$ in Eq. (20) conform to the physics that the pressure of the relatively non-wetting phasefluid is greater than or equal to that in the relatively wetting phasefluid. Otherwise, the fluid of the non-wetting phase dissolves into the wetting phase; or equivalently, the pressure of the dissolved non-wetting phase-fluid must be smaller than that of the wetting phasefluid to prevent its bubbling into a non-wetting phase.

The relationship between the degrees of saturation and unsaturated hydraulic conductivity for multiphase flow system proposed by Parker et al. (1987) is given as

$$k_{r1} = S_{t1}^{\frac{1}{2}}\left[1 - \left(1 - S_{t1}^{\frac{1}{m}}\right)^{m}\right]^{2} \qquad (21)$$

$$k_{r2} = (S_{t2} - S_{t1})^{\frac{1}{2}}\left[\left(1 - S_{t1}^{\frac{1}{m}}\right)^{m} - \left(1 - S_{t2}^{\frac{1}{m}}\right)^{m}\right]^{2} \qquad (22)$$

$$k_{r3} = (1 - S_{t2})^{\frac{1}{2}}\left(1 - S_{t2}^{\frac{1}{m}}\right)^{2m} \qquad (23)$$

where k_{r1}, k_{r2}, and k_{r3} individually represent the relative permeability of water, NAPL and air phases.

5. RESULTS AND DISSCUSSION

Seven examples are illustrated to test the plausibility and completeness of the proposed model that is analogous to van Genuchten's renowned parametric model. The hypothetical scaling factors and curve shape parameters for all examples are given in Table 1. In addition, Table 2 exhibits the given pressure head of each phase-fluid in each example. Note that Examples 1 - 3 are the systems with three-phase-fluids, while Examples 4 - 7 are with four-phasefluids.

For Examples 1 through 3, the Parker et al.'s model and the present model are both employed to obtain the degrees of saturation given the pressure heads. For Examples 4 through 7, only the present model is used to simulate the degrees of saturation given the pressure heads since Parker et al.'s model does not include the systems of more than three-phasefluids. The simulated degrees of saturation are given in Table 3.

Discussion and results are presented with example problems below.

Example 1: The results obtained with Parker et al.'s model are obviously unreasonable that a negative degree of saturation exists in the NAPL phasefluid (note that in the model Fluid 1 is water, Fluid 2 is NAPL, and Fluid 3 is air). The reason is that the products of the scaling factor and capillary pressure head do not satisfy the constraint in Eq. (15). The detail of calculation and assessment is shown as follows.

$$\overline{\alpha}_{21} h_{c21} = 2.0448 \text{ and } \overline{\alpha}_{32} h_{c32} = 2.1651 \Rightarrow (\overline{\alpha}_{21} h_{c21}) < (\overline{\alpha}_{32} h_{c32}) \tag{24}$$

which does not satisfy the constraint in Eq. (15) resulted from Parker et al.'s model. This implies that such pressure head distributions are not considered feasible using the Parker et al.'s model. On the other hand, the results obtained with the proposed model are in the range of feasible solutions.

Example 2: In this example, the difference between $(\overline{\alpha}_{21} h_{c21})$ and $(\overline{\alpha}_{32} h_{c32})$ is further increased in contrast to that in Example 1 as

$$\overline{\alpha}_{21} h_{c21} = 2.0448 \text{ and } \overline{\alpha}_{32} h_{c32} = 4.0596 \Rightarrow (\overline{\alpha}_{21} h_{c21}) \ll (\overline{\alpha}_{32} h_{c32}) \tag{25}$$

Compared Example 2 with Example 1, as the difference between $(\overline{\alpha}_{21} h_{c21})$ and $(\overline{\alpha}_{32} h_{c32})$ is increased, the degree of saturation for NAPL fluid in Table 3 becomes more negative (-0.2008 for Example 2 versus -0.0210 for Example 1) and hence more unreasonable. In contrast, the results with the proposed model are considered more reasonable. Furthermore, the difference in the degree of saturation between Fluids 3 and 2 for this example is greater than that for Example 1, as expected.

Example 3: In this system of three-phasefluids, the products of the scaling factor and capillary pressure head for $(\overline{\alpha}_{21} h_{c21})$ and $(\overline{\alpha}_{32} h_{c32})$ satisfy the constraint of Eq. (15). Because of this, Parker et al.'s model also yields plausible simulations. Naturally, the present model also yields plausible simulations. As to which model would give more reasonable results, only extensive calibrations and validations can resolve the question, which is beyond the scope of this paper and, of course, is not the objective of this paper.

Table 1. Scaling factors and curve shape parameters for all seven examples.

Example	$\overline{\alpha}_{21}$ (cm^{-1})	$\overline{\alpha}_{32}$ (cm^{-1})	$\overline{\alpha}_{43}$ (cm^{-1})	N
1	0.11	0.099	NA	2.2
2	0.11	0.099	NA	2.2
3	0.11	0.099	NA	2.2
4	0.11	0.12	0.099	2.2
5	0.11	0.12	0.099	2.2
6	0.11	0.12	0.099	2.2
7	0.11	0.12	0.099	2.2

Table 2. The given pressure head of each phase-fluid for all seven examples.

Example	h_1 (cm)	h_2 (cm)	h_3 (cm)	h_4 (cm)
1	979.91	998.50	1020.37	NA
2	979.91	998.50	1039.50	NA
3	977.31	998.09	1020.37	NA
4	978.13	998.09	1005.33	1020.37
5	979.91	998.50	998.50	1020.37
6	979.91	979.91	979.91	1020.37
7	978.13	998.09	1020.37	1020.37

Table 3. Calculated degrees of saturation of each phase for all seven examples.

Example	S_1	S_2	S_3	S_4	Remark
1-1	0.3825	-0.0210	0.6389	NA	Parker et al.'s Model
1-2	0.1381	0.2230	0.6389	NA	Present Model
2-1	0.3825	-0.2008	0.6389	NA	Parker et al.'s Model
2-2	0.0695	0.1122	0.8184	NA	Present Model
3-1	0.3417	0.0127	0.6456	NA	Parker et al.'s Model
3-2	0.1211	0.2333	0.6456	NA	Present Model
4	0.1353	0.2446	0.1332	0.4869	Present Model
5	0.1381	0.2230	0.0000	0.6389	Present Model
6	0.1845	0.0000	0.0000	0.8155	Present Model
7	0.1031	0.1864	0.7105	0.0000	Present Model

Example 4: The simulated degrees of saturation for this system of four-phasefluids are: $S_1 = 0.1353$, $S_2 = 0.2446$, $S_3 = 0.1332$, and $S_4 = 0.4869$. These results seem plausible. The present model generates positive degrees of saturation because the physically mandated constraint of $p_{c\alpha+1,\alpha} \geq 0$ is satisfied. The mandate requires that the pressure head of relatively non-wetting phasefluid is greater than or equal to that of a relatively wetting phasefluid.

Example 5: In this example, the pressure heads of Fluid 2 and Fluid 3 are identical as shown in Table 2. Thus the capillary pressure head h_{c32} is equal to zero. This implies that Fluid 3 should vanish in such pressure head distributions. Indeed, it is seen that the simulated degree of saturation for Fluid 3 is zero (Table 3), and Fluid 3 disappears as expected.

Example 6: In this example, the pressure heads for Fluids 1 through 3 are identical as shown in Table 2. Thus, the number of zero capillary pressure head for this system of four-phasefluids is increased to two, $h_{c21} = h_2 - h_1 = 0$ and $h_{c31} = h_3 - h_2 = 0$. Physically, this system of four-phasefluids is reduced to a system of two-phasefluids. The solutions of the degree of saturations are $S_1 = 0.1845$, $S_2 = 0$, $S_3 = 0$, and $S_4 = 0.8155$ (Table 3). The degrees of saturation in two fluids are equal to zero, implying that the system is reduced to two fluids from four fluids, as expected.

Example 7: Different from Examples 5 and 6, the zero capillary pressure head occurs between Fluids 3 and 4 in this example because the pressure heads in these two fluids are identical (Table 2). Simulations with the proposed model, Eq. (16), are listed in the last row of Table 3. It is seen that the degree of saturation of Fluid 4 vanishes in this example. As a result, the relatively non-wetting phase, Fluid 4, disappears because of $S_4 = 0$. Whenever, the pressure heads between two contacting fluids are identical, the relatively non-wetting phase disappears. This is so because the non-wetting fluid dissolves into the wetting phase fluid.

6. CONCLUSION

Simultaneous measurements of the degree of saturation and capillary pressure heads in multiple-phasefluids have been a challenge. Therefore, it is necessary to develop plausible analytic models describing the constitutive relation of saturation and capillary pressure head for systems of multiple-phasefluids.

In this paper, a postulation is made that the capillary pressure between two contacting fluids in a multiple phase-fluid system is a unique function of a single variable of grouped degrees of saturation. This single variable is the ratio of the accumulated degree of saturation up to the wetting phase to that up to the corresponding non-wetting phase. Based on this fundamental postulation, a closed-form expression of the saturation-capillary pressure relationship for a system of L-phasefluids is proposed analogously to that of van Genuchten's renowned model. For a two-phasefluids system, the model is reduced to that of van Genuchten's model in formality, but it has quite a different physical interpretation. Through both physical arguments and mathematical manipulations, the undue constraint of the currently popular three phase-fluid model is alleviated with the proposed model. Seven examples are employed to demonstrate the plausibility and completeness of the model. Through these examples, the model is demonstrated to pose no limitations on the distributions of pressure in all phases except for the physical requirement that pressure in the non-wetting phase must be greater or equal to that in the wetting phase. The model provides a feasible generalized saturation-capillary pressure relationship for systems of arbitrary number of phasefluids.

Acknowledgements Research is supported by National Science Council under Contract No. NSC 99-2116-M-008-020 with National Central University.

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